Problem Set 2 Solutions

1. Graph a typical indifference curve for the following utility functions and determine whether they obey the assumption of diminishing $MRS$:

a. $U(x, y) = 3x + y$

Since the indifference curves are not bowed towards the origin, they do not obey the assumption of diminishing $MRS$.

b. $U(x, y) = \sqrt{xy}$

Since the indifference curves are bowed towards the origin, they do obey the assumption of diminishing $MRS$. Alternatively, we know $MU_x$ and $MU_y$ are both positive. So when quantity of $X$ increases, quantity of $Y$ must decrease. The $MRS_{xy} = \frac{Y}{X}$. So as $X$ increase, the denominator gets bigger and $MRS$ decreases. As $X$ increase, $Y$ decreases and the numerator gets smaller so $MRS$ decreases. Both these effects work so that as $X$ increase $MRS$ decreasing.
c. \( U(x,y) = \sqrt{x^2 + y^2} \)

Since the indifference curves are not bowed towards the origin, they do not obey the assumption of diminishing MRS. Alternatively, we know MUx is positive and MUy is positive, so as quantity of X increases, the quantity of Y must decrease.

The MRS_{xy} = \frac{X}{Y}. So as X increases the numerator increases so MRS increases. As X increases Y decreases so denominator is getting smaller and MRS increases. Both these effects work in the same direction so MRS is increasing not diminishing.

d. \( U(x,y) = \sqrt{x^2 - y^2} \)

This one is a bit trickier because the indifference curves are not downward sloping. We want to check whether it is the case that as \( x \) increases, \( MRS_{xy} \) decreases. We begin by calculating \( MRS_{xy} :\)
\[
MRS_{x,y} = \frac{MU_x}{MU_y} \\
= \frac{x(x^2 - y^2)^{1/2}}{y(x^2 - y^2)^{1/2}} \\
= -\frac{x}{y}
\]

The next thing we do is substitute for \( y \) using the equation of the indifference curve so as to have \( MRS_{x,y} \) expressed solely in terms of \( x \).

The equation of the indifference curve is
\[
\overline{U} = \sqrt{x^2 - y^2},
\]
where \( \overline{U} \) represents a constant level of utility. Solving this equation for \( y \) gives us
\[
\overline{U}^2 = x^2 - y^2 \\
y^2 = x^2 - \overline{U}^2 \\
y = \sqrt{x^2 - \overline{U}^2}
\]

Substituting for \( y \) in our expression for \( MRS_{x,y} \) yields
\[
MRS_{x,y} = -\frac{x}{y} \\
= -\frac{x}{\sqrt{x^2 - \overline{U}^2}}
\]

As \( x \) approaches \( \overline{U} \), the denominator goes to 0, which means that \( MRS_{x,y} \) goes to \(-\infty\). However, as \( x \) goes to \( \infty \), \( MRS_{x,y} \) goes to \(-1\). So, as \( x \) increases, \( MRS_{x,y} \) becomes larger.

Hence, \( MRS_{x,y} \) is not diminishing.
e. \( U(x, y) = x^{2/3} y^{1/3} \)

![Graph showing indifference curve with \( U = 8 \)]

Since the indifference curves are bowed towards the origin, they do obey the assumption of diminishing MRS.

f. \( U(x, y) = \log x + \log y \)

Note: “\( \log \)” was intended to represent the natural logarithm (i.e., “\( \log \)” = “\( \ln \)”).

![Graph showing indifference curve with \( U = 1 \)]

Since the indifference curves are bowed towards the origin, they do obey the assumption of diminishing MRS.
2. Suppose a consumer’s preferences for two goods can be represented by the Cobb-Douglas utility function \( U(x, y) = A x^\alpha y^\beta \), where \( A, \alpha, \) and \( \beta \) are positive constants.

a. What is \( MRS_{x,y} \)?

We begin by calculating the marginal utilities with respect to \( x \) and \( y \):

\[
MU_x = \frac{\partial U(x, y)}{\partial x} = \alpha A x^{\alpha-1} y^\beta
\]

\[
MU_y = \frac{\partial U(x, y)}{\partial y} = \beta A x^\alpha y^{\beta-1}
\]

We can then use these marginal utilities to obtain \( MRS_{x,y} \):

\[
MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{\alpha A x^{\alpha-1} y^\beta}{\beta A x^\alpha y^{\beta-1}}
\]

\[
= \frac{\alpha}{\beta} \frac{x}{y}
\]
b. Is $MRS_{x,y}$ diminishing, constant, or increasing as the consumer substitutes $x$ for $y$ along an indifference curve?

To determine this, we need to substitute for $y$ using the equation of the indifference curve so as to have $MRS_{x,y}$ expressed solely in terms of $x$.

The equation of the indifference curve is

$$\bar{U} = Ax^\alpha y^\beta,$$

where $\bar{U}$ represents a constant level of utility. Solving this equation for $y$ gives us

$$y^\beta = \frac{\bar{U}}{Ax^\alpha},$$

$$y = \left( \frac{\bar{U}}{Ax^\alpha} \right)^{\frac{1}{\beta}},$$

$$y = \frac{1}{\bar{U}^\frac{1}{\beta}} \left( A \right)^\frac{1}{\beta} \left( x^\beta \right)^\alpha.$$

Substituting for $y$ in our expression for $MRS_{x,y}$ yields

$$MRS_{x,y} = \frac{\alpha}{\beta} \frac{y}{x}$$

$$= \frac{\alpha}{\beta} \frac{1}{x} \left( \frac{\bar{U}^\frac{1}{\beta}}{\left( A \right)^\frac{1}{\beta} \left( x^\beta \right)^\alpha} \right)$$

$$= \frac{\alpha}{\beta} \left( \frac{\bar{U}}{A} \right)^\frac{1}{\beta} \left( \frac{1}{x^\frac{\alpha}{\beta}} \right).$$

Since $A$, $\alpha$, and $\beta$ are positive constants, the first two terms in the equation above are also positive and constant. Moreover, the exponent on $x$, $1 + \frac{\alpha}{\beta}$, is also positive and constant. Therefore, as $x$ increases, $MRS_{x,y}$ decreases.

That is, $MRS_{x,y}$ is diminishing.
c. On a graph with $x$ on the horizontal axis and $y$ on the vertical axis, draw a typical indifference curve. Indicate on your graph whether the indifference curve will intersect either or both axes.

We know “more is better” because $MU_x$ and $MU_y$ are both positive; therefore, the indifference curves must be downward sloping. Moreover, we determined in part b that $MRS_{x,y}$ is diminishing; therefore, the indifference curves must be bowed in towards the origin.

And finally, recall that the equation of a typical indifference curve is given by

$$U = A x^\alpha y^\beta,$$

where $U$ represents a constant level of utility. Since for any $U > 0$, it cannot be the case that either $x$ or $y$ equals zero, the indifference curves do not intersect either axis.

These three observations indicate that the indifference map must be as follows:
3. Ch 3, Problem 3.6

For the following sets of goods draw two indifference curves, $U_1$ and $U_2$, with $U_2 > U_1$. Draw each graph placing the amount of the first good on the horizontal axis.

a. Hot dogs and chili (the consumer likes both and has a diminishing marginal rate of substitution of hot dogs for chili)

![Indifference curve for hot dogs and chili]

b. Sugar and Sweet’N Low (the consumer likes both and will accept an ounce of Sweet’N Low or an ounce of sugar with equal satisfaction)

![Indifference curve for sugar and Sweet’N Low]
c. Peanut butter and jelly (the consumer likes exactly 2 ounces of peanut butter for every ounce of jelly)

![Graph of Peanut Butter and Jelly Preferences]

d. Nuts (which the consumer neither likes nor dislikes) and ice cream (which the consumer likes)

![Graph of Nuts and Ice Cream Preferences]
e. Apples (which the consumer likes) and liver (which the consumer dislikes)

4. Repeat Problem 2 for the quasi-linear utility function $U(x, y) = 2\sqrt{x} + y$.

We begin by calculating the marginal utilities with respect to $x$ and $y$:

$$MU_x = \frac{\partial U(x, y)}{\partial x} = \frac{1}{\sqrt{x}}$$

$$MU_y = \frac{\partial U(x, y)}{\partial y} = 1$$

We can then use these marginal utilities to obtain $MRS_{x,y}$:

$$MRS_{x,y} = \frac{MU_x}{MU_y} = \frac{1}{\sqrt{x}}$$

Unlike Problem 2, this problem does not require us to substitute for $y$ in order to determine whether or not $MRS_{x,y}$ is diminishing. (The reason is that $y$ does not appear in our expression for $MRS_{x,y}$.) We can just look at our expression for $MRS_{x,y}$ and see that as $x$ increases, $MRS_{x,y}$ decreases. Therefore, $MRS_{x,y}$ is diminishing.

To summarize, we know “more is better” because $MU_x$ and $MU_y$ are both positive; therefore, the indifference curves must be downward sloping. Moreover, we determined that $MRS_{x,y}$ is diminishing; therefore, the indifference curves must be bowed in towards the origin.
And finally, note that the equation of a typical indifference curve is given by

\[ \overline{U} = 2\sqrt{x} + y, \]

where \( \overline{U} \) represents a constant level of utility. Unlike the indifference curves we examined in Problem 2, the indifference curve represented by the equation \( \overline{U} = 2\sqrt{x} + y \) do allow for \( x \) or \( y \) to equal zero: for any \( \overline{U} > 0 \), we can either set \( x = 0 \) (in which case \( y = \overline{U} \)) or set \( y = 0 \) (in which case \( x = \overline{U}^2 / 4 \)). Therefore, the indifference curves in this problem intersect both axes.

These three observations indicate that the indifference map must be as follow.

5. Julie has preferences for food, \( F \), and clothing, \( C \), are derived by a utility function \( U(F,C) = FC \). Food costs $1 a unit and clothing costs $2 a unit. Julie has $12 to spend on food and clothing.

a. Give the equation for Julie’s budget line. What is the slope of the budget line?

Julie’s budget equation is: \( F + 2C = 12 \), slope = \( -\frac{P_F}{P_C} = -2 \)

b. Graph Julie’s budget line. Place food on the vertical axis and clothing on the horizontal axis.

c. On the same graph, draw an indifference curves that is tangent to his budget line.
d. Julie is a utility maximize, write the objective function.

e. Write down the full optimization problem with the objective function and the constraint.

Her objective function is her utility function \( U(F,C) = FC \)

d. Using calculus and algebra, find the basket of food and clothing that maximizes Julie's utility (i.e. solve the maximization problem you wrote down in e) (Assume Julie can purchase fractional amounts of both goods.)

\[
\begin{align*}
\max_{C,F} & \quad U(F,C) = FC \\
\text{s.t.} & \quad F + 2C = 12
\end{align*}
\]

Step 1: using the budget constraint solve for \( F \) or \( C \): \( F = 12 - 2C \)

Step 2: Substitute the budget constraint into the utility function so the utility function is a function of one good.

\[
\begin{align*}
\max_C & \quad U(C) = (12 - 2C)C = 12C - 4C^2
\end{align*}
\]

Step 3: Now we need to maximize \( U \) with respect to \( C \).

\[
\begin{align*}
\frac{\partial U}{\partial C} &= 12 - 4C = 0 \\
C &= 3
\end{align*}
\]

Step 4: Sub \( C \) back into budget constraint to figure out optimal \( F \)

\[
F + 2(3) = 12 \quad \text{so} \quad F = 6
\]

The basket which optimizes Julie’s utility function is \( C = 3 \) and \( F = 6 \).
6. Each day Peter, who is in the third grade, eats lunch at school. He only likes liver ($L$) and onions ($N$), and these provide him a utility of
\[ U(L, N) = \ln(LN). \]
Liver costs $4.00 per serving, onions cost $2.00 per serving, and Peter’s mother gives him $8.00 to spend on lunch.

a. Give the equation for Peter’s budget line. What is the slope of the budget line?

Peter’s budget line is $4L + 2N = 8$. The slope of the budget line is $-\frac{P_N}{P_L} = -\frac{1}{2}$.

b. Graph Peter’s budget line. Place the number of liver servings on the vertical axis and the number of onion servings on the horizontal axis.

c. On the same graph, draw several of Peter’s indifference curves, including one that is tangent to his budget line.
d. Using calculus and algebra, find the basket of liver and onions that maximizes Peter's utility. (Assume Peter can purchase fractional amounts of both goods.) Mark this basket on your graph.

**REMEMBER YOU MAY NOT SOLVE THE PROBLEM THE WAY THE BOOK DOES, YOU MUST SET UP THE MAXIMIZATION PROBLEM AND SOLVE THAT**

1. Set up the Maximization Problem

\[ \max_{L,N} U(L, N) = \ln(LN) \]

\[ s.t. \quad 4L + 2N = 8 \]
2. Rewrite budget constraint so \( N \) is in terms of \( L \) and \( L \) in terms of \( N \)

\[
N = 4 - 2L.
\]

3. Sub the rewritten budget constraint back into the objective function and rewrite the maximization.

\[
\max_{L} U(L) = \ln \left( L^* (4 - 2L) \right)
\]

4. Maximize the utility function with respect to \( L \). To do this you need to find the slope of the utility function (take the derivative w.r.t \( L \)) and set the slope equal to 0. Hint before you do this you want to expand the utility function to be \( \ln(4L - 2L^2) \).

\[
\frac{\partial U}{\partial L} = \frac{1}{L(4-2L)} (4-4L) = 0
\]

\[
\frac{4-4L}{L(4-2L)} = 0
\]

\[
4-4L = 0
\]

\[
L^* = 1
\]

5. To find \( N^* \), sub \( L^* \) back into the budget constraint.

\[
N = 2 - 2(1) = 2 - 2
\]

\[
N^* = 2
\]

Therefore the optimal bundle is \( (L^*, N^*) = (1, 2) \).

Therefore, the optimal basket is \( (L^*, N^*) = (1, 2) \). The optimal basket is marked in the figure below:
7. Ch 4, Problem 4.4

Jane likes hamburgers ($H$) and milkshakes ($M$). Her indifference curves are bowed in and toward the origin and do not intersect the axes. The price of a milkshake is $1 and the price of a hamburger is $3. She is spending all her income at the basket she is currently consuming, and her marginal rate of substitution of hamburgers for milkshakes is 2. Is she at an optimum? If so, show why. If not, should she buy fewer hamburgers and more milkshakes, or the reverse?

Note: here we have to use the tangency condition since we are not given enough information for consumer optimization.

From the given information, we know that $P_H = 3$, $P_M = 1$, and $MRS_{H,M} = 2$. Comparing the $MRS_{H,M}$ to the price ratio,

$$MRS_{H,M} = 2 < \frac{P_H}{P_M} = \frac{3}{1}.$$

Since these are not equal, Jane is not currently at an optimum. In addition, we can say that

$$\frac{P_H}{P_M} > MRS_{H,M} = \frac{MU_H}{MU_M},$$

which is equivalent to

$$\frac{MU_M}{P_M} > \frac{MU_H}{P_H}.$$

That is, the “bang for the buck” from milkshakes is greater than the “bang for the buck” from hamburgers. So Jane can increase her total utility by reallocating her spending to purchase fewer hamburgers and more milkshakes.
8. Ch 4, Problem 4.7

Toni likes to purchase round trips between the cities of Pulmonia and Castoria and other goods out of her income of $10,000. Fortunately, Pulmonian Airways provides air service and has a frequent-flyer program. A round trip between the two cities normally costs $500, but any customer who makes more than 10 trips a year gets to make additional trips during the year for only $200 per round trip.

a. On a graph with round trips on the horizontal axis and “other goods” on the vertical axis, draw Toni’s budget line. (Hint: This problem demonstrates that a budget line need not always be a straight line.)

Let $Y$ represent the quantity of “other goods” Toni purchases and let $P_Y$ represent the price of a unit of “other goods.”

If Toni spends her entire income of $10,000 on other goods, she will obtain $10,000/P_Y$ units of other goods. This gives us the $Y-$ intercept of the budget line. But what happens if Toni spends her entire income on round trip flights ($X$)?

Toni has to pay $500 per flight for the first 10 flights that she buys, so those 10 flights will cost her $5,000, leaving her with $5,000. How many flights can she buy with the $5,000 she has left over? After the tenth flight, she can buy additional flights at $200 per flight. So, she can buy $5,000/200 = 25$ additional flights at $200 a piece.

In sum, if Toni spends her entire income of $10,000 on flights, she will obtain $10 + 25 = 35$ flights. This gives us the $X-$ intercept of the budget line.

But what happens in between the two intercepts? We know that the slope of the budget line is $-P_X/P_Y$. So, when $0 \leq X \leq 10$, the slope of the budget line is $-500/P_Y$, but when $10 < X \leq 35$, the slope of the budget line is $-200/P_Y$. The change in slope generates the “kink” in the budget line at $X = 10$. 

![Budget Line Diagram](image-url)
b. On the graph you drew in part a, draw a set of indifference curves that illustrates why Toni may be better off with the frequent-flyer program.

With the indifference curves drawn on the above graph, Toni is better off with the frequent flyer program than she would be without it. If there were no frequent flyer program, the top portion of the budget line would extend straight down to the $X$ axis instead of becoming flatter towards the bottom, and Toni would be unable to afford the optimal bundle shown in the graph above.

c. On a new graph draw the same budget line you found in part a. Now draw a set of indifference curves that illustrates why Toni might not be better off with the frequent-flyer program.

With the indifference curves drawn on the graph above, Toni is no better off with the frequent flyer program than she would be without it. If there were no frequent flyer program, the top portion of the budget line would extend straight down to the $X$ axis instead of becoming flatter towards the bottom – but given the indifference curves drawn, this wouldn’t affect Toni’s choice of bundle.